数理解析研究所講究録
第 1932 巻 2015 年 21-24

Dynamical decomposition theorems

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Abstract. In this article, we study some dynamical decomposition theorems of spaces related to given homeomorphisms. First, we introduce new notions of 'bright spaces' and 'dark spaces' of homeomorphisms except n times, and by use of the notions we show some dynamical decomposition theorems of spaces related to given homeomorphisms. Next, we show that if $f: X \to X$ is a homeomorphism of an n-dimensional separable metric space X with zero-dimensional set of periodic points, then X can be decomposed into a zero-dimensional bright space of f except n times and an $(n-1)$-dimensional dark space of f except n times, and also by use of dark spaces, we can show some decomposition theorems of X related to dimension theory and dynamical systems. Finally, we study dynamical decompositions of continuum-wise expansive homeomorphisms.

1 Introduction

In this article, we assume that all spaces are separable metric spaces and dimension means the topological dimension dim. Also, let $\mathbb{N}$ and $\mathbb{Z}$ denote the set of natural numbers and the set of integers, respectively. If A is a subset of a space X, then $\mathrm{cl}(A)$, $\mathrm{bd}(A)$ and $\mathrm{int}(A)$ denote the closure, the boundary and the interior of A in X, respectively. For a collection $\mathcal{G}$ of subsets of X,

$$\mathrm{ord}(\mathcal{G}) = \sup\{\mathrm{ord}_x(\mathcal{G}) |\ x \in X\},$$

where $\mathrm{ord}_x(\mathcal{G})$ is the number of members of $\mathcal{G}$ which contains x.

We introduce new notions of 'bright spaces' and 'dark spaces' of homeomorphisms except n times, and by use of the notions we prove some dynamical decomposition theorems of spaces related to given homeomorphisms. For a homeomorphism $f: X \to X$ of a space X and $k \in \mathbb{N}$, let $P_k(f)$ denote the set of points of period $\leq k$. Also, $P(f)$ denotes the set of all periodic poins of f. A subset Z of X is a *bright space* of f except n times ($n \in \{0\} \cup \mathbb{N}$) if for any $x \in X$,

$$|\{p \in \mathbb{Z} |\ f^p(x) \notin Z\}| \leq n,$$

where $|A|$ denotes the cardinality of a set A. Also we say that $L = X - Z$ is a *dark space* of f except n times. Note that for any $x \in X$, $|O_f(x) \cap L| \leq n$, where $O_f(x) = \{f^p(x) |\ p \in \mathbb{Z}\}$ denotes the orbit of x, and also note that $L \cap P(f) = \phi$. For a dark space L of f except n times and $0 \leq j \leq n$, we put

$$A_f(L, j) = \{x \in X |\ |\{p \in \mathbb{Z} |\ f^p(x) \in L\}| = j\}\ (= \{x \in X |\ |O_f(x) \cap L| = j\}).$$

$A_f(L,j)$ denotes the set of all point $x \in X$ whose orbit $O_f(x)$ appears in L just j times. Note that $P(f) \subset A_f(L,0)$ and $A_f(L,j)$ is f-invariant, i.e. $f(A_f(L,j)) = A_f(L,j)$ and $A_f(L,i) \cap A_f(L,j) = \phi$ if $i \neq j$. Hence we have the f-invariant decomposition related to the dark space L as follows;

$$X = A_f(L,0) \cup A_f(L,1) \cup \cdots \cup A_f(L,n).$$

2 Dynamical decomposition theorems of homeomorphisms with zero-dimensional sets of periodic points

It is well-known that a space X has at most dimension n ($n \in \{0\} \cup \mathbb{N}$) (i.e. $\dim X \leq n$) if and only if X can be represented as a union of $(n+1)$ zero-dimensional subspaces of X (see [2, 12]). The following proposition may be known.

Proposition 2.1. *Suppose that X is a space with $\dim X = n$ $(< \infty)$ and $f : X \to X$ is a homeomorphism. Then there exist f-invariant zero-dimensional dense G_δ-sets $A_f(j)$ $(j = 0, 1, 2, ..., n)$ of X such that*

$$X = A_f(0) \cup A_f(1) \cup \cdots \cup A_f(n).$$

In [1], Arts, Fokkink and Vermeer proved the following interesting theorem of dynamical systems of homeomorphisms under some dimensional conditions of periodic points.

Theorem 2.2. ([1, Theorem 8]) *Suppose that $f : X \to X$ is a homeomorphism of a (metric) space X with $\dim X \leq n$ $(< \infty)$. Then there exists a dense G_δ-set Z of X such that $\dim Z = 0$ and*

$$X = Z \cup f(Z) \cup f^2(Z) \cup \cdots \cup f^n(Z)$$

if and only if $\dim P_k(f) < k$ for each $1 \leq k \leq n$.

In this article, under the condition of $\dim P(f) \leq 0$, we prove more chaotic decomposition theorems of dynamical systems of homeomorphisms. In [3, 4, 5, 8, 9], we studied some dynamical properties of homeomorphisms with zero-dimensional set of periodic points. Now, we need the following lemma.

Lemma 2.3. (cf. [4, Lemma 3.5] and [3, Lemma 2.2]) *Suppose that X is a space with $\dim X = n$ $(< \infty)$ and $f : X \to X$ is a homeomorphism with $\dim P(f) \leq 0$. Let F be an F_σ-set of X with $\dim F \leq 0$. Then for each $j \in \mathbb{N}$, there is a locally finite countable open cover $\mathcal{C}(j) = \{C(j)_\alpha \mid \alpha \in \mathbb{N}\}$ of X such that*
(1) $\mathrm{mesh}(\mathcal{C}(j)) < 1/j$,
(2) $\mathrm{ord}(\mathcal{G}) \leq n$, *where* $\mathcal{G} = \{f^p(\mathrm{bd}(C(j)_\alpha)) \mid \alpha \in \mathbb{N},\ j \in \mathbb{N}$ *and* $p \in \mathbb{Z}\}$ *and*
(3) $F \cap L = \phi$, *where* $L = \cup\{(\mathrm{bd}(C(j)_\alpha)) \mid \alpha \in \mathbb{N},\ j \in \mathbb{N}\}$.

The following theorem is a key result.

Theorem 2.4. *Suppose that X is a space with $\dim X = n$ $(< \infty)$ and $f : X \to X$ is a homeomorphism. Then there exists a bright space Z of f except n times such that Z is a zero-dimensional dense G_δ-set of X and the dark space $L = X - Z$ of f is a $(n-1)$-dimensional F_σ-set of X if and only if $\dim P(f) \leq 0$.*

Corollary 2.5. *Suppose that X is a space with $\dim X = n$ $(< \infty)$ and $f : X \to X$ is a homeomorphism. Then there exists a zero-dimensional G_δ-dense set Z of X such that for any $(n+1)$ integers $k_0 < k_1 < \cdots < k_n$,*

$$X = f^{k_0}(Z) \cup f^{k_1}(Z) \cup \cdots \cup f^{k_n}(Z)$$

if and only if $\dim P(f) \leq 0$.

Theorem 2.6. *Suppose that X is a space with $\dim X = n$ $(< \infty)$ and $f : X \to X$ is a homeomorphism with $\dim P(f) \leq 0$. If L is a dark space of f except n times such that L is an F_σ-set of X and $\dim\ (X - L) \leq 0$, then $\dim A_f(L, j) = 0$ for each $j = 0, 1, 2, ..., n$. In particular, there is the f-invariant zero-dimensional decomposition of X related to the dark space L:*

$$X = A_f(L, 0) \cup A_f(L, 1) \cup \cdots \cup A_f(L, n).$$

Finally, as a special case we consider the case that $f : X \to X$ is a continuum-wise expansive homeomorphism of a compact metric space X. A homeomorphism $f : X \to X$ of a compact metric space (X, d) is *expansive* (see [11]) if there is $c > 0$ such that for any $x, y \in X$ with $x \neq y$, there is an integer $k \in \mathbb{Z}$ such that $d(f^k(x), f^k(y)) \geq c$. Similarly, a homeomorphism $f : X \to X$ of a compact metric space (X, d) is *continuum-wise expansive* (see [6, 7]) if there is $c > 0$ such that for any nondegenerate subcontinuum A of X, there is an integer $k \in \mathbb{Z}$ such that diam $f^k(A) \geq c$. Note that every expansive homeomorphism is continuum-wise expansive. Such $c > 0$ is called an *expansive constant* for f. It is known that if a compact metric space X admits a continuum-wise expansive homeomorphism f on X, then $\dim X < \infty$ and every minimal set of f is zero-dimensional (see [11] and [6]). Moreover, $\dim I_0(f) \leq 0$, where

$$I_0(f) = \bigcup\{M |\ M \text{ is a zero-dimensional } f\text{-invariant closed set of } X\}$$

(see [7, Proposition 2.5]). In particular, $\dim P(f) \leq 0$. We need the following proposition.

Proposition 2.7. ([6, Proposition 5.1]) *Suppose that $f : X \to X$ is a homeomorphism of a compact metric space X. Then the following are equivalent.*
(1) *f is continuum-wise expansive.*
(2) *There is $\delta > 0$ such that if $\mathcal{C}$ is any finite open cover of X with* $\mathrm{mesh}(\mathcal{C}) < \delta$ *and any $\gamma > 0$, there is a sufficiently large natural number N such that if $A, B \in \mathcal{C}$, each component of $f^{-n}(\mathrm{cl}(A)) \cap f^n(\mathrm{cl}(B))$ has diameter less than γ for each $n \geq N$.*

In the case of continuum-wise expansive homeomorphisms, by use of compact dark spaces we obtain the following decomposition theorem.

Theorem 2.8. *Suppose that X is a compact metric space with* $\dim X = n$ $(< \infty)$ *and $f : X \to X$ is a continuum-wise expansive homeomorphism. Then there exists a compact $(n-1)$-dimensional dark space L of f except n times such that* $\dim A_f(L, j) = 0$ *for each* $j = 0, 1, 2, ..., n$. *In particular, there is the f-invariant zero-dimensional decomposition of X related to the compact dark space L:*

$$X = A_f(L, 0) \cup A_f(L, 1) \cup \cdots \cup A_f(L, n).$$

Remark. (1) In Theorem 2.8, the bright space $Z = X - L$ of f is open in X and n-dimensional. (2) In Theorem 2.8, suppose that $\dim X = 1$. Then L is a compact zero-dimensional dark space of f except 1 time such that $\dim A_f(L, j) = 0$ for each $j = 0, 1$ if and only if L is a zero-dimensional compactum such that $f^i(L) \cap L = \emptyset$ for any $i \in \mathbb{N}$ and dim $(X - \cup_{i \in \mathbb{Z}} f^i(L)) = 0$.

Example. Let $f : I = [0, 1] \to I$ be the 'tent' map of the unit interval I defined by $f(x) = 2x$ for $0 \leq x \leq 1/2$ and $f(x) = 2 - 2x$ for $1/2 \leq x \leq 1$. Consider the inverse limit

$$X = \{(x_i)_{i=1}^{\infty} \in I^{\infty} \mid f(x_{i+1}) = x_i \text{ for } i \in \mathbb{N}\} \subset I^{\infty}$$

of f and the shift map $\tilde{f} : X \to X$ defined by $\tilde{f}((x_i)_{i=1}^{\infty}) = (f(x_i))_{i=1}^{\infty}$. Then $\tilde{f}$ is a continuum-wise exapnsive homeomorphism of the Knaster continuum X. Consider the subset

$$L = \{(x_i)_{i=1}^{\infty} \in X \mid x_1 = 1\}.$$

Then we can easily see that L is a zero-dimensional compactum (in fact, a Cantor set) such that $\tilde{f}^i(L) \cap L = \phi$ for any $i \in \mathbb{N}$ and dim $(X - \cup_{i \in \mathbb{Z}} \tilde{f}^i(L)) = 0$ and hence L is a compact zero-dimensional dark space L of $\tilde{f}$ except 1 time such that $\dim A_{\tilde{f}}(L, 0) = 0$. In fact, $X = A_{\tilde{f}}(L, 0) \cup A_{\tilde{f}}(L, 1)$ is a zero-dimensional decomposition of the Knaster continuum X.

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